

The Holographic Superconductor Vortex

Marc Montull,^{1,*} Alex Pomarol,^{1,†} and Pedro J. Silva^{1,2,‡}

¹*Departament de Física and IFAE, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona*

²*Institut de Ciències de l'Espai (CSIC) and Institut d'Estudis Espacials de Catalunya (IEEC/CSIC),
Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona*

A gravity dual of a superconductor at finite temperature has been recently proposed. We present the vortex configuration of this model and study its properties. In particular, we calculate the free energy as a function of an external magnetic field, the magnetization and the superconducting density. We also find the two critical magnetic fields that define the region in which the vortex configurations are energetically favorable.

INTRODUCTION

The Gauge/Gravity duality, that relates strongly interacting gauge theories to theories of gravity in higher dimensions, has opened a new window to study many different strongly interacting systems. The applicability of this approach is very vast ranging from particle physics to plasma and nuclear physics. In Ref. [1] a model for a dual description of a superconductor was proposed. The model showed to have a critical temperature T_c under which the system goes into a superconducting phase. The properties of this phase have been thoroughly studied [2], showing a resemblance with those of a Type II superconductor. In spite of this, Abrikosov vortices, known to happen in Type II superconductors, have not yet been obtained. The purpose of this letter is to show that in this type of gravity duals vortex solutions indeed exist and can be energetically favorable in the presence of external magnetic fields. Due to the nonlinear nature of these configurations, we will have to rely on numerical methods. Among other physical properties, we will calculate the free energy and the range of the magnetic field $B_{c1} \leq B \leq B_{c2}$ at which the superconductor is at the intermediate phase (Shubnikov phase) characterized by vortex configurations. Further aspects of these solutions will be presented elsewhere.

THE MODEL

The physical system to study is a conformal strongly coupled superconductor in 3D at finite temperature and charge density. Its gravitational dual theory [1] is an asymptotically AdS-Schwarzschild space-time in 4D. The gravitational degrees of freedom are coupled to an U(1) gauge field A_μ and a complex scalar Ψ . The action that summarizes the above model is given by

$$S = \int d^4x \sqrt{-G} \left\{ \frac{1}{16\pi G_N} (R + \Lambda) - \frac{1}{g^2} \mathcal{L} \right\},$$

with $\mathcal{L} = \frac{1}{4} F^2 + \frac{1}{L^2} |D_\mu \Psi|^2 + \frac{m^2}{L^4} |\Psi|^2$. (1)

G_N is the 4D gravitational Newton constant, the cosmological constant Λ defines the asymptotic AdS radius L via the relation $\Lambda = -3/L^2$ and $D_\mu = \partial_\mu - iA_\mu$. We use the convention where the metric G has signature $(-, +, +, +)$, with coordinates (t, z, r, ϕ) where t is time, z is the holographic direction such that the AdS-boundary occurs at $z = 0$, and (r, ϕ) are polar coordinates parameterizing the remaining 2D plane. For the scalar mass m^2 we will focus on two possible values: $m^2 = -2, 0$. Other values are expected to give similar behaviors [3].

We will work in the so-called probe approximation, where the gravity sector is effectively decoupled from the matter sector and therefore, there is no back-reaction on the background metric due to \mathcal{L} . This regime is achieved in the limit of large g , when compared to the gravitational strength. In this limit we can, without loss of generality, fix $g = 1$. In our conventions, the background AdS-Schwarzschild Black hole (BH) metric is given by

$$ds^2 = \frac{L^2}{z^2} (-f(z)dt^2 + dr^2 + r^2 d\phi^2) + \frac{L^2}{z^2 f(z)} dz^2, \quad (2)$$

where $f(z) = 1 - (z/z_h)^3$.

As we are considering the theory at finite temperature, we have to take the Euclidian regime with compact time $it \in [0, 1/T]$ where $T = 3/(4\pi z_h)$. Therefore, the holographic coordinate runs from the AdS-boundary at $z = 0$ to the BH horizon at $z = z_h$. Notice that we work with a planar BH with energy per unit area $\varepsilon = L^2/(8\pi G_N z_h^3)$. Then, the AdS/CFT duality tells us that the above are precisely the temperature and energy density of the dual superconductor.

The gauge field has the usual AdS-boundary behavior

$$A_\nu \rightarrow a_\nu + J_\nu z, \quad (3)$$

where $a_\nu = (\mu, a_i)$ corresponds to the potentials on the dual CFT, while $J_\nu = (-\rho, J_i)$ plays the role of the conjugated currents. We will consider the case in which the charge density ρ is fixed constant. The other potentials a_i are related to turning on either electromagnetic fields or sample velocities in the dual CFT, depending on the interpretation we give to the AdS/CFT duality. The first interpretation is what we will use in this article, while the

second one is relevant for superfluids¹ [4]. Similarly, the scalar field has the following AdS-boundary behavior

$$|\Psi| \rightarrow az^{3-\Delta} + bz^\Delta, \quad (4)$$

where $\Delta = 2, 3$ (for $m^2 = -2, 0$) corresponds to the dimension of the dual operator \mathcal{O}_Δ responsible for the U(1) breaking, and b determines the vacuum expectation value of this operator. The value of a corresponds to an explicit breaking of the U(1) symmetry and will then be turned to zero². Having fixed m^2 , the only parameters of the model are the scales T and $\sqrt{\rho}$.

It has been reported in Ref. [1, 3] that for $\rho \neq 0$ the system undergoes a phase transition at

$$\begin{aligned} T_c &\simeq 0.12\sqrt{\rho} \quad \text{for } m^2 = -2, \\ T_c &\simeq 0.09\sqrt{\rho} \quad \text{for } m^2 = 0, \end{aligned} \quad (5)$$

where the two phases are related to a charged BH and a charged BH with a non-trivial scalar hair. At $T < T_c$, the system is at the hairy phase corresponding to a superconducting phase. In Refs. [5, 6] the model was also studied in the presence of an external magnetic field B using a dyonic BH with a probe scalar field. The result was a bounded superconducting region or drop, that squeezes to zero size as we increase B . The above suggested that we are dealing with a Type II superconductor. If this is the case, Abrikosov vortex configurations should be present in this model.

We stress that, as is usual in this approach, we are treating the electromagnetic field of the 3D dual theory as a nondynamical background. This corresponds to take the 3D electric charge $e \rightarrow 0$, while keeping constant B and ρ .

THE VORTEX SOLUTION

We use the Ansatz given by

$$\Psi = \psi(r, z) e^{in\phi}, \quad A_0 = A_0(r, z), \quad A_\phi = A_\phi(r, z), \quad (6)$$

with all other fields set to zero. This Ansatz preserves global U(1) transformations when combined with a rotation in the 2D plane. The fields A_r, A_z can be consistently set to zero since our Ansatz fulfills $\partial_r \text{Arg}[\Psi] = \partial_z \text{Arg}[\Psi] = 0$. The winding number $n \in \mathbb{Z}$ determines different topological solutions. With the above Ansatz we

obtain from Eq. (1) the following equations of motion:

$$\begin{aligned} z^2 \partial_z \left(\frac{f}{z^2} \partial_z \psi \right) + \frac{1}{r} \partial_r (r \partial_r \psi) \\ + \left(\frac{A_0^2}{f} - \frac{(A_\phi - n)^2}{r^2} - \frac{m^2}{z^2} \right) \psi = 0, \\ \partial_z (f \partial_z A_\phi) + r \partial_r \left(\frac{1}{r} \partial_r A_\phi \right) - \frac{2\psi^2}{z^2} (A_\phi - n) = 0, \\ f \partial_z^2 A_0 + \frac{1}{r} \partial_r (r \partial_r A_0) - \frac{2\psi^2}{z^2} A_0 = 0. \end{aligned} \quad (7)$$

In order to describe a dual superconductor at fixed ρ in the presence of an external magnetic field B , the AdS/CFT correspondence tells us that we must impose the AdS-boundary conditions

$$\psi|_{z=0} = 0, \quad \partial_z A_0|_{z=0} = -\rho, \quad A_\phi|_{z=0} = \frac{1}{2} r^2 B, \quad (8)$$

for the case $m^2 = 0$, while for $m^2 = -2$ the first condition must be $\partial_z \psi|_{z=0} = 0$ (this is equivalent to set $a = 0$ in Eq. (4)). At the horizon $z = z_h$ we require the field configurations to be regular; in particular we set $A_0|_{z=z_h} = 0$ as usual, to have a well-defined Euclidean continuation. Similar reasoning at $r = 0$ implies that for $n \neq 0$

$$\psi|_{r=0} = 0, \quad \partial_r A_0|_{r=0} = 0, \quad A_\phi|_{r=0} = 0, \quad (9)$$

while for $n = 0$, $\partial_r \psi|_{r=0} = 0$. We will be considering a 3D superconductor of radius R that we will take to be much bigger than the vortex radius. This is implemented by setting a nonzero ρ extending from $r = 0$ to $r = R$.

The 2D system of the three partial differential equations of Eq. (7) is nonlinear, and therefore requires to be solved numerically. For this purpose we have used the COMSOL 3.4 package [7]. In our numerical studies we have chosen

$$R = \frac{50}{\sqrt{\rho}}, \quad T = 0.065\sqrt{\rho}. \quad (10)$$

This corresponds to

$$\frac{T}{T_c} \simeq 0.74 \quad (0.54), \quad (11)$$

for the case of $m^2 = 0$ (-2).

In Fig. 1 we show the order parameter $\langle \mathcal{O}_\Delta \rangle = \frac{1}{\Delta} z^{1-\Delta} \partial_z \psi|_{z=0}$ of the dual superconductor. We can see that this goes to zero at the origin where the vortex is placed. For the value of the magnetic field, we have chosen

$$B_n = \frac{2n}{R^2}, \quad (12)$$

corresponding to the value at which the magnetic flux crossing a surface of constant z , $\Phi = \int d\phi \int_0^R r dr B$, equals $2\pi n$. This is the quantized flux going through the n -vortex of the dual superconductor.

¹ In fact, the vortex solution we present in this article can be identified with vortex configurations in a superfluid, once the appropriated reinterpretations are made.

² For the case $m^2 = -2$ there is the possibility to have $b = 0$ and $a \neq 0$ corresponding to have a dual CFT operator of dimension one [2].

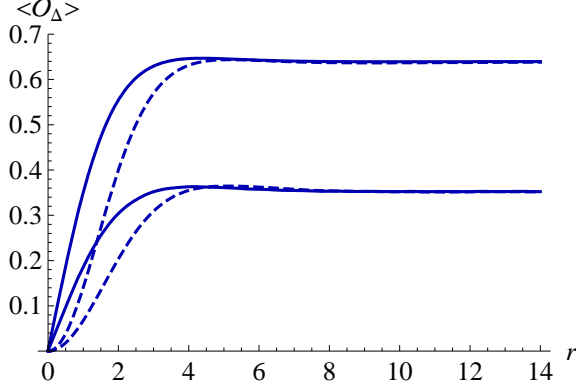


FIG. 1: Order parameter $\langle \mathcal{O}_\Delta \rangle$ for the $n = 1$ (solid) and $n = 2$ (dashed) vortex configuration. The lower (upper) curves correspond to the case $m^2 = 0$ (-2). Presented in units of $\sqrt{\rho} = 1$.

FREE ENERGY, MAGNETIZATION AND CRITICAL MAGNETIC FIELDS

We are interested to determine the free energy of the superconductor configurations with $n = 0, 1, 2$ to know which one is energetically favorable as we vary B . By the AdS/CFT, the free energy F of the superconductor is given by

$$\frac{F[T, B, \rho]}{T} = S_E + \frac{\pi}{T} \int_0^R dr r A_0 \partial_z A_0 \Big|_{z=0}, \quad (13)$$

where the right-hand side is evaluated on-shell in the 4D theory with the boundary conditions given in Eq. (8). The second term of Eq. (13) has been added to guarantee the variational principle when working at fixed $\partial_z A_0$ on the AdS-boundary. Since, as we will see, the phase transition to vortex configurations occurs at small values of B , we can treat the magnetic field as a small perturbation and separate the solution as

$$\psi \rightarrow \psi + \delta\psi, \quad A_0 \rightarrow A_0 + \delta A_0, \quad A_\phi \rightarrow A_\phi + \delta A_\phi, \quad (14)$$

where the unperturbed solution (ψ, A_0, A_ϕ) corresponds to that at zero external magnetic field, i.e., $A_\phi|_{z=0} = 0$, while the perturbation $(\delta\psi, \delta A_0, \delta A_\phi)$ must fulfill

$$\delta A_\phi|_{z=0} = \frac{1}{2} r^2 B, \quad \partial_z \delta A_0|_{z=0} = 0, \quad \delta\psi|_{z=0} = 0, \quad (15)$$

for $m^2 = 0$ and $\partial_z \delta\psi|_{z=0} = 0$ for $m^2 = -2$. By integrating by parts the free energy of the n -vortex configuration can be written, up to B^2 terms, as

$$F_n(B) \simeq F_n(0) - \alpha_n B + \frac{1}{2} \beta_n B^2, \quad (16)$$

where we have defined

$$\begin{aligned} F_n(0) &= 2\pi \int_0^R dr \int_0^{z_h} dz \frac{r}{z^2} \left(\frac{A_0^2}{f} - \frac{A_\phi(A_\phi - n)}{r^2} \right) \psi^2 \\ &\quad - \pi \int_0^R dr r A_0 \partial_z A_0 \Big|_{z=0}, \\ \alpha_n &= \frac{2\pi}{B} \int_0^R \frac{dr}{r} \delta A_\phi \partial_z A_\phi \Big|_{z=0}, \\ \beta_n &= -\frac{2\pi}{B^2} \int_0^R \frac{dr}{r} \delta A_\phi \partial_z \delta A_\phi \Big|_{z=0}. \end{aligned} \quad (17)$$

Notice that the positive-defined quantities α_n and β_n do not depend on B , since $\delta A_\phi \propto \delta A_\phi|_{z=0} \propto B$. Eq. (16) has a simple interpretation in terms of the magnetization M of the superconductor. Using $M = -\partial F / \partial B$, we can write

$$F_n(B) = F_n(0) - \int_0^B M_n dB, \quad (18)$$

where the magnetization of the n -vortex configuration M_n in the z -component is given by

$$M_n = \frac{1}{2} \int d\phi dr r (\vec{r} \times \vec{J})_z = \pi \int dr r J_\phi. \quad (19)$$

From the AdS/CFT dictionary, we have that

$$\langle J_\phi \rangle = -\frac{\delta F}{\delta A_\phi|_{z=0}} = \partial_z A_\phi + \partial_z \delta A_\phi \Big|_{z=0}, \quad (20)$$

that together with Eq. (19) leads to our final expression for the magnetization

$$M_n = \alpha_n - \beta_n B. \quad (21)$$

Using this expression into Eq. (18), we recover the free energy of Eq. (16).

For the free energy at $B = 0$ we obtain

$$F_n(0) \simeq F_0(0) + 0.9(1.5)n^2 \ln[R\rho^{1/2}] \sqrt{\rho} + c_n, \quad (22)$$

where $c_0 = 0$, $c_1 \simeq 1.2(3.7)\sqrt{\rho}$, $c_2 \simeq 0.3(4)\sqrt{\rho}$ and

$$F_0(0) \simeq 5(4)R^2 \rho \sqrt{\rho}, \quad (23)$$

for the case $m^2 = 0(-2)$. This shows that, as expected, the vortex configurations have for $B = 0$ a larger energy than the $n = 0$ solution. Note that $F_0(0)$ grows with the volume of the superconductor ($\propto R^2$), although not the difference $F_{1,2}(0) - F_0(0)$ that is only logarithmically sensitive to R for $R \rightarrow \infty$, as expected for 3D vortices in the absence of electromagnetic fields. For the magnetization we find

$$\alpha_n \simeq 0.4(0.7) n R^2 \sqrt{\rho}, \quad \beta_n \simeq 0.05(0.09) R^4 \sqrt{\rho}. \quad (24)$$

From Eq. (16) it is clear that there is a critical value for B at which the difference between the free energies $F_1(B) - F_0(B)$ is zero. This value is usually referred as B_{c1} and marks the beginning of the mixed phase where the magnetic field starts to penetrate the superconductor. For the case of $m^2 = 0$ we have

$$\frac{F_1(B) - F_0(B)}{\sqrt{\rho}} \simeq 0.9 \ln[R\rho^{1/2}] + 1.2 - 0.8 \frac{B}{B_1}, \quad (25)$$

that for $R = 50/\sqrt{\rho}$ equals to zero at

$$B_{c1} \simeq 6B_1, \quad (26)$$

where B_1 is defined in Eq. (12). For $m^2 = -2$ we get similar values, $B_{c1} \simeq 7B_1$. At higher magnetic field values than B_{c1} the vortex configuration is preferred. Notice that for $R \rightarrow \infty$, we have $B_1 \rightarrow 1/R^2$ and therefore $B_{c1} \rightarrow 0$, indicating that the non-vortex solution is never favorable at any $B \neq 0$.

For the configuration with $n = 2$, we find that its free energy is less than that for $n = 0, 1$ if $B \gtrsim 10(14)B_1$ for $m^2 = 0(-2)$. At this high magnetic field, however, we expect that the free energy of a solution with two $n = 1$ vortices will be energetically more favorable, as it happens in Type II superconductors. Indeed, for two vortices sufficiently separated we expect

$$F(B) \simeq F_0(0) + 2[(F_1(0) - F_0(0)) - \alpha_1 B] + E_{\text{int}} + \frac{1}{2}\beta_1 B^2, \quad (27)$$

where E_{int} is the interaction energy between the two vortices. Therefore the difference between the free energy of two $n = 1$ vortices and one $n = 2$ vortex goes as $\Delta F \simeq E_{\text{int}} - 1.8(3) \ln[R\rho^{1/2}]\sqrt{\rho}$ for $m^2 = 0(-2)$. As a consequence a configuration with two $n = 1$ vortices will be preferred for $E_{\text{int}} < 1.8(3) \ln[R\rho^{1/2}]\sqrt{\rho}$ that is expected for a large superconductor.

On the other hand, as B increases from B_{c1} , a configuration with more and more vortices is expected to be favorable, until we reach a certain critical value B_{c2} at which there is another phase transition; for $B > B_{c2}$ the normal phase is preferred. We estimate this value by the magnetic field at which the superconducting region of the $n = 0, 1$ configurations shrink to zero size. We find $B_{c2} \simeq 3(5)\rho$ for $m^2 = 0(-2)$.

In Fig. 2 we plot the values of the free energy as a function of B for the configurations $n = 0, 1, 2$ from the exact numerical solutions. We can see that the critical magnetic values at which the lines cross are similar to the approximate ones given above.

Finally, we calculate the “superconducting density” $n_s(r)$ defined as

$$n_s(r) = \langle J_\phi J^\phi \rangle = \frac{\delta F}{\delta A_\phi^2|_{z=0}} = - \left. \frac{\partial_z \delta A_\phi}{\delta A_\phi} \right|_{z=0}, \quad (28)$$

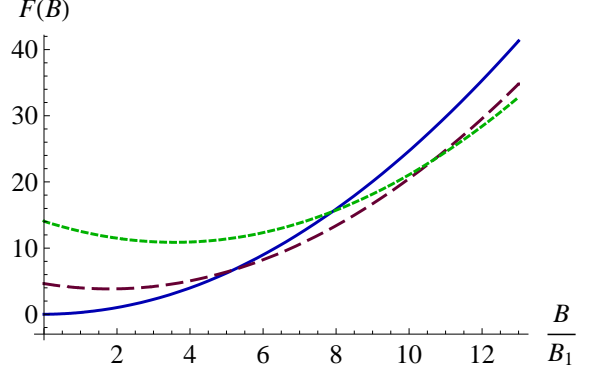


FIG. 2: Free energy for the $m^2 = 0$ case as a function of the external magnetic field for the $n = 0$ (solid), $n = 1$ (dashed) and $n = 2$ (dotted) vortex configuration. Presented in units of $\sqrt{\rho} = 1$.

where in the last equality we have used Eq. (16). In Fig. 3 we show $n_s(r)$ for the different configurations. We notice that the vortex configuration fullfills $\langle J_\phi \rangle = -n_s(r)(\delta A_\phi|_{z=0} - n)$, as expected from a spontaneously broken $U(1)$ symmetry. For a non-vortex configuration the superconducting density is constant $n_s(r) \simeq 0.28(0.48)\sqrt{\rho}$ for $m^2 = 0(-2)$. This determines the penetration length $\lambda = 1/(e\sqrt{n_s})$ where e is the electric charge of the dual superconductor.

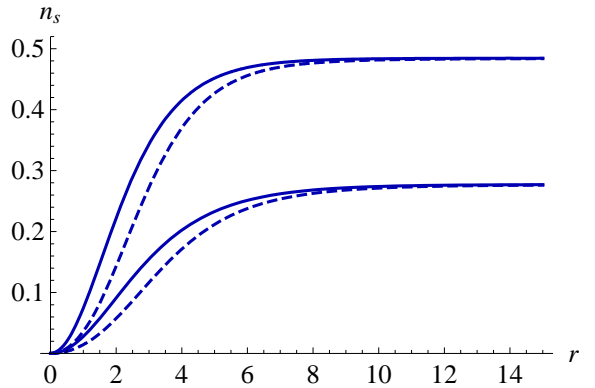


FIG. 3: Superconducting density $n_s(r)$ for the $n = 1$ (solid) and $n = 2$ (dashed) vortex configuration. The lower (upper) curves correspond to the case $m^2 = 0$ (-2). Presented in units of $\sqrt{\rho} = 1$.

Note Added: While finishing this paper, we learned of Ref. [8] which has also studied the vortex solution in holographic superconductors.

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* Electronic address: mmontull@ifae.es

† Electronic address: alex.pomarol@uab.cat

‡ Electronic address: psilva@ifae.es

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